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Optimal growth and the golden rule in a two-sector model of capital accumulation

Mehdi Senouci*

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Abstract

We contribute to the literature on optimal growth in two-sector models by solving a Ramsey problem with a concave utility function. The unique possible steady-state is independent of initial conditions and of the instantaneous utility function, but not of the discount rate, and is characterized by a wage-rental ratio depending solely on the technology of the capital sector. For an initially low-capital economy, we show that the wage-rental ratio increasingly converges to its balanced value during transition. If the consumption sector is relatively capital-intensive, the relative price of capital increases during transition. If the investment sector is relatively more capital-intensive, it decreases. We also prove that a negative shock on the subjective rate of impatience, that makes the social planner more patient, leads to an immediate positive jump in asset prices.

Keywords: Capital accumulation, optimal growth, golden rule, two-sector models.
JEL codes: C61, C62, E13, E21, E22, O11, O41.

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Comments are very welcome.

Introduction

Two-sector growth models that explicitly distinguish between the sector producing consumption goods and the one producing investment goods, each being endowed with a specific production function, were pretty in fashion in the 1960's, but sound quite old-world today. Their dismiss from the standard tool-box of growth theorists is first of all a consequence of the revolution in growth theory starting with the Solow-Swan model. The property that the saving rate cannot affect the long-run growth rate but only steady-state consumption *level* led to a gradual but massive loss of interest in the role of capital in the process of growth. But another cause of this retirement of the two-sector framework has been one unpleasant property of early models: in Meade (1961), Uzawa (1961, 1963) and Kurz (1963), the system is unstable when the investment sector is more capital-intensive than the consumption sector. This conclusion consistently casted doubt on the legitimacy of the overall approach. As pointed out by Solow's (1961), it is indeed very hard to accept a model that depends on such a strong and arbitrary assumption[1].

However, this complication arose solely as a consequence of the not-less arbitrary hypotheses – in the form of defined saving rates out of labor and capital income – taken in the early literature. In the subsequent literature on optimal growth in two-sector models, such a pathology does not appear: in both cases, the optimal path follows a saddle-path, however of different form depending on which sector is relatively capital-intensive[2].

Two-sector models are useful because they add a 'capital' degree of freedom to the baseline neoclassical growth framework, and allows to analyze the pattern of the price of capital during the process of accumulation of capital, as well as to gain some deeper insight about consumption and investment in the economy. Furthermore, the one-sector problem can be seen as a reduced two-sector model where the two sectors share the same capital-intensity[3], which makes the latter models natural extensions of the formers.

Capital was important in the early ages of growth, but it would also be erroneous to think that it has no special place in today's seemingly-capitalized economies. As figure 1 shows, the deflator of private investment[4] has decreased relatively to the deflator of GDP and to the CPI in the USA between 1982 and 2010, meaning that investment goods have seen their price fall relatively to that of consumption goods. This fact, not to mention the recurrence of bubbles or the coincidence of real and financial cycles, suggests that the past thirty years have seen the emergence of a new structural model of growth where *capital,* in the broadest sense, has a new role. Indeed, the data seem to militate unequivocally for the thesis of structural change in 1980-82 and, before that, at the end of the Bretton Woods era[5].

This paper contributes to the literature on capital and growth by solving, for the first time, the Ramsey problem with a concave utility function in a two-sector environment with no technical progress. Previous articles on the same subject[6] implicitly suppose that the utility function is linear in consumption so that, to our knowledge, no solution of the model presented below exists.

[1] Solow (1961) also gives a plain analysis of this instability property.

[2] Uzawa (1965), Shell (1967). Srinivasan (1964) takes the problem from the point of view of the optimal saving rate, but only considers the case where the consumption sector is relatively capital-intensive. See also the "Sceptical notes on Uzawa..." by Haque (1970).

[3] This is also true of models that suppose that there is only one final good, which is readily transformable in A units of the capital good after each period.

[4] I.e., the ratio of nominal to real investment. All figures are relegated to last section.

[5] See Clarida, Galí and Gertler (2000) and Galí and Gambetti (2009) on the former, and Greenwood and Yorukoglu (1997) on the latter.

[6] Srinivasan (1964), Uzawa (1965) and Shell (1967).

We will successively describe the steady-state and the pattern of optimal transition, to finally investigate into the effect of shocks on the rate of preference for present on the dynamics of the system.

The main results are the following. There is a unique steady-state corresponding to a 'quasi-golden rule' similar to the one-sector version of Cass (1965). The steady-state balanced wage-rental ratio is determined solely by the technology of the *investment* sector, a property for which we give an interpretation. We then take the case of an economy that is initially undercapitalized and show that, then, the wage-rental ratio *increases* during transition. The relative price of capital in terms of the consumption good *increases* (resp. *decreases*) along the optimal path if the consumption sector is relatively *more* (resp. *less*) capital-intensive than the investment sector. An unexpected shock that brings the rate of impatience down leads to an immediate positive jump of the relative price of capital in terms of consumption.

The rest of the paper is organized as follows. Section 1 presents the model and introduces necessary notations, as well as the capital-intensity assumption. Section 2 describes the unique steady-state and the golden rules. Section 3 sets up the dynamic system. Section 4 analyzes the pattern of transition of an economy that has initially less capital than is desired in the long-term, in the case the consumption sector is always relatively capital-intensive. Section 5 makes the same for the case it is the investment sector that is always relatively capital-intensive. Section 6 analyzes the dynamical effects of shocks on the subjective discount rate. Section 7 concludes.

1 An exact two-sector Ramsey model

1.1 Presentation of the problem

Time evolves discretely. There are two homogenous goods and two sectors: the consumption sector produces the consumption good (C) and the investment sector produces the investment good (I). Both sectors hire capital and labor, which are perfectly mobile across industries, and produce output according to constant returns to scale production functions, respectively F and G for consumption and investment:

$$\forall t, \quad \begin{aligned} C_t &= F(K_t^C, L_t^C) \\ I_t &= G(K_t^I, L_t^I) \end{aligned} \tag{1}$$

We suppose that F and G are twice continuously differentiable, display diminishing marginal returns to inputs and satisfy the standard Inada conditions.

Workforce is constant at L and capital stock evolves according to the equation:

$$K_{t+1} = (1-\delta)K_t + I_t, \tag{2}$$

where δ is the constant depreciation rate. Initial capital endowment is exogenously given at K_0.

Population is made of a unique infinitely-lived representative agent[7] with a twice continuously differentiable instantaneous utility function $v(C_t) = Lu\left(\frac{C_t}{L}\right)$, (so that for all C, $v'(C) = u'(C/L) = u'(c)$) and discounting future utility by a constant factor β. We suppose that v – and, thus, u – satisfy traditional Inada conditions[8]. The social planner solves the following problem:

$$\begin{array}{rl} max & \sum_{t=0}^{\infty} \beta^t v\left(C_t\right) \\ C_t & \leq F\left(K_t^C, L_t^C\right) \\ I_t & \leq G\left(K_t^I, L_t^I\right) \\ K_t^C + K_t^I & \leq K_t \\ L_t^C + L_t^I & \leq L \\ K_{t+1} & = (1-\delta)K_t + I_t \end{array} \tag{3}$$

Let the vector of Lagrangian multipliers associated with the five constraints be $(p^C,\ p^I,\ r,\ w,\ q)_t$. The Lagrangian is:

$$\begin{array}{rl} \mathcal{L} = \sum_{t=0}^{\infty} \beta^t & \left[v\left(C_t\right) - p_t^C\left(C_t - F\left(K_t^C, L_t^C\right)\right) - p_t^I\left(I_t - G\left(K_t^I, L_t^I\right)\right)\right. \\ & +w_t\left(L_t - L_t^C - L_t^I\right) + r_t\left(K_t - K_t^C - K_t^I\right) \\ & \left.-q_t\left(K_{t+1} - (1-\delta)K_t - I_t\right)\right] \end{array} \tag{4}$$

As v satisfies the Inada condition when $C \to 0$, the social planner will never choose a zero level of consumption. Therefore, for all $t \geq 0$, $C_t \neq 0$, and first order conditions are:

$$v'\left(C_t\right) = p_t^C \tag{5}$$

$$\left\{ \begin{array}{l} q_t \leq p_t^I \\ q_t = p_t^I \quad \textit{if and only if} \quad I_t > 0 \end{array} \right. \tag{6}$$

$$p_t^C \frac{\partial F}{\partial L_t^C} = w_t \tag{7}$$

[7] We would of course obtain the same results in the corresponding perfect competitive framework – i.e. with fair rental price for labor and capital, fair price for both types of output and eventually a perfect credit market – with infinitely-lived agents.

[8] As said before, to our knowledge, all existing two-sector optimal growth models assume linear utility. This has the consequence of setting the implicit value of the consumption good at constant level, and is sometimes legitimized by taking the consumption good as the 'numéraire'. For instance, we reach conclusions that are somewhat different than those of Shell (1967), which performs a similar analysis. In Shell (1967), specialization in the production of investment goods can occur for the initially very capital-poor economies because the intertemporal benefits of producing capital increase when the level of capital become low. Here, because of the Inada conditions imposed on the utility function, consumption never falls to zero. Our results are smoother, but not qualitatively different.

$$\begin{cases} p_t^I \frac{\partial G}{\partial L_t^I} \leq w_t \\ p_t^I \frac{\partial G}{\partial L_t^I} = w_t \quad \textit{if and only if} \quad L_t^I > 0 \end{cases} \tag{8}$$

$$p_t^C \frac{\partial F}{\partial K_t^C} = r_t \tag{9}$$

$$\begin{cases} p_t^I \frac{\partial G}{\partial K_t^I} \leq r_t \\ p_t^I \frac{\partial G}{\partial K_t^I} = r_t \quad \textit{if and only if} \quad K_t^I > 0 \end{cases} \tag{10}$$

$$q_t = \beta(r_{t+1} + (1-\delta)q_{t+1}) \tag{11}$$

(5) states that the price of the consumption good is equal to marginal utility of consumption. Thus p^C is expressed in terms of utility per unit of consumption. (6) states that the supply price of capital p^I must be equal to the demand price q, except at the corner solution where production of investment good is zero, in which case the shadow supply price exceeds demand price. Equations from (7) to (10) reflect equalization of marginal productivity of labor and capital to their rental prices in both sectors[9], except in the case of corner solutions where $I = 0$. Remark that (8) and (10) prove that p^I is expressed in terms of utility per unit of investment good. r and w are respectively in terms of utility per unit of capital and per unit of labor.

Finally, (11) reflects the fairness of the intertemporal pricing of capital. The first-best subjective value of one unit of capital at t is equal to the discounted value of the reward it will bring tomorrow, including capital gains $(1-\delta)q_{t+1}$.

It is more convenient to work with per-capita variables. Let k be the macroeconomic capital-labor ratio, k^C and k^I be the sectoral capital-labor ratios and l^C and l^I be the share of labor employed in each sector:

$$k_t = \frac{K_t}{L}, \quad k_t^C = \frac{K_t^C}{L_t^C}, \quad k_t^I = \frac{K_t^I}{L_t^I} \tag{12}$$

$$l_t^C = \frac{L_t^C}{L}, \quad l_t^I = \frac{L_t^I}{L} \tag{13}$$

We apply the usual transformation to the production functions:

$$f(k^C) = F(k^C, 1), \quad g(k^I) = G(k^I, 1) \tag{14}$$

[9] I.e. respectively to the wage rate w and to the capital rent r.

If both goods are produced, and dropping the time argument, first-order conditions from (7) to (10) can be reduced to:

$$p^C\left(f(k^C)-k^Cf'(k^C)\right)=p^I\left(g(k^I)-k^Ig'(k^I)\right)=w \tag{15}$$

$$p^Cf'(k^C)=p^Ig'(k^I)=r \tag{16}$$

Notice that whenever labor and capital are at full employment, which is here clear at any date, we have:

$$\begin{cases} l^Ck^C+l^Ik^I=k \\ l^C+l^I=1 \end{cases} \tag{17}$$

So that whenever $k^C\neq k^I$:

$$l^C=\frac{k-k^I}{k^C-k^I},\quad l^I=\frac{k^C-k}{k^C-k^I} \tag{18}$$

1.2 Preliminary notations

Let $\omega=w/r$ be the wage-rental ratio. When both goods are produced, (15) and (16) imply that, independently of the relative price of investment and consumption goods, technological marginal rates of substitution are equalized across sectors at optimum:

$$\forall t,\ \frac{f(k^C)}{f'(k^C)}-k^C=\frac{g(k^I)}{g'(k^I)}-k^I=\omega, \tag{19}$$

For all $\omega\geq 0$, there exists unique values of k^C and k^I satisfying (19). Differentiating (19), we obtain that:

$$\frac{\partial k^C}{\partial\omega}\left(-\frac{ff''}{f'^2}\right)\left(k^C(\omega)\right)=\frac{\partial k^I}{\partial\omega}\left(-\frac{gg''}{g'^2}\right)\left(k^I(\omega)\right)=1, \tag{20}$$

so that the functions $k^C(\omega)$ and $k^I(\omega)$ are increasing in the wage-rental ratio. We also have that $k^C(\omega)\to_0 0$ and $k^I(\omega)\to_0 0$.

We can now introduce the formal relative capital intensity assumptions[10]:

Definition 1.1. *(Capital intensity assumption)*

1. *The consumption industry is more capital-intensive than the investment industry if, for all* $\omega > 0$,
$$k^C(\omega) > k^I(\omega);$$

2. *The investment industry is more capital-intensive than the consumption industry if, for all* $\omega > 0$,
$$k^C(\omega) < k^I(\omega).$$

Relative prices of inputs and of goods

Let p_t be the relative price of investment goods in terms of consumption goods at t :

$$p_t = \frac{p_t^I}{p_t^C} \tag{21}$$

From the demand side, according to (5) and (6), when both goods are produced, we have that:

$$p_t = \frac{q_t}{u'(c_t)}, \tag{22}$$

where $c_t = C_t/L$.

In optimal one-sector models, the shadow value of consumption and the shadow value of capital are necessarily equal: when there is only one good, at optimum its marginal value for one use (consumption) equals its marginal value for the other use (investment)[11].

When both goods are produced, in virtue of (16), it holds that:

$$p(\omega) = \frac{f'(k^C(\omega))}{g'(k^I(\omega))} \tag{23}$$

[10] We do not attempt to solve the problem in presence of capital-intensity reversals for some wage-rental ratios.

[11] The theoretical structure of two-good models fundamentally differ from that of one-good models. In the latter class of models, the act of saving is *technically* the same as the act of investment: what is produced but not consumed is saved and increases the capital stock, which can itself be consumed as well. In models of the former type, equilibrium on the market for consumption goods does not mechanically lead to equilibrium on the market for investment goods. The deeper difference of the two approaches has to do with the interpretation of the $S = I$ equation.

Theorem 1.1. *(Stolper-Samuelson-Uzawa) For all ω at the optimum and if both goods are produced, the logarithmic derivative of the relative price of the capital good in terms of the consumption good is given by:*

$$\frac{1}{p}\frac{dp}{d\omega} = \frac{1}{k^I(\omega)+\omega} - \frac{1}{k^C(\omega)+\omega}, \tag{24}$$

which is positive if the consumption sector is relatively more capital-intensive, and negative if the investment sector is relatively more capital-intensive.

So there is an instantaneous monotonic relation between the wage-rental ratio and the relative price of output of both industries. In the case the consumption sector is more capital-intensive, a relative increase of wages leads to an increase in the relative supply cost of the investment good in terms of the consumption good, so that $p(\omega) = p^I/p^C$ increases. The opposite is true in the case the investment sector is more capital intensive.

Critical prices and patterns of specialization

Finally, we present the critical input and output relative prices, via which we will define the specialization frontiers, above and under which occur specialization of production in any of the sectors. Define, for each k, the two critical wage-rental ratios $\omega^C(k)$ and $\omega^I(k)$ by:

$$\omega^C(k) = \frac{f}{f'}(k) - k, \quad \omega^I(k) = \frac{g}{g'}(k) - k \tag{25}$$

It is immediate to prove that ω^C and ω^I are increasing, and that:

$$\lim_{k\to 0} \omega^C(k) = \lim_{k\to 0} \omega^I(k) = 0 \tag{26}$$

$$\lim_{k\to \infty} \omega^C(k) = \lim_{k\to \infty} \omega^I(k) = \infty \tag{27}$$

and that the following proposition holds:

Proposition 1.1. *1. If the consumption sector is relatively capital-intensive, then for all $k > 0$, $\omega^C(k) < \omega^I(k)$;*

2. If the investment sector is relatively capital-intensive, then for all $k > 0$, $\omega^C(k) > \omega^I(k)$.

We now define the minimum and maximum wage-rental ratios by:

$$
\begin{aligned}
\omega_{min}(k) &= \min\left\{\omega^C(k), \omega^I(k)\right\} \\
\omega_{max}(k) &= \max\left\{\omega^C(k), \omega^I(k)\right\},
\end{aligned} \tag{28}
$$

and the critical price ratios by:

$$
\begin{aligned}
p_{min}(k) &= \min\left\{p^C(k), p^I(k)\right\} \\
p_{max}(k) &= \max\left\{p^C(k), p^I(k)\right\},
\end{aligned} \tag{29}
$$

where $p^C(k) = p(\omega^C(k)) = \frac{f'(k^C(\omega^C(k))}{g'(k^I(\omega^C(k))}$ and $p^I(k) = p(\omega^I(k)) = \frac{f'(k^C(\omega^I(k))}{g'(k^I(\omega^I(k))}$. Because of theorem 1.1 and proposition 1.1, we have that for each $k > 0$, $p_{min}(k) = p^C(k) < p_{max}(k) = p^I(k)$.

From (25) we see that the functions $\omega^C(k)$ and $\omega^I(k)$ are increasing with respect to k. Therefore, $\omega_{min}(k)$ and $\omega_{max}(k)$ are also increasing. $p_{min}(k)$ and $p_{max}(k)$ are both increasing in the case where the consumption sector is capital-intensive, and both decreasing in the case the investment sector is capital-intensive.

The critical relative prices p_{min} and p_{max} determine the frontiers of specialization in the (k, p) plan.

When relative price p is less than $p_{min}(k)$, no investment good is produced as the first unit of investment good would be too costly to produce in relation to the demand price.

In the hypothetical case where p would be greater than $p_{max}(k)$, no consumption good would be produced as consumption would not be valued enough or, equivalently, the last unit of investment good that is produced when all scarce resources are put in the investment sector brings more intertemporal utility than the first unit of consumption. This case does not arise at optimum because the marginal value of consumption goes to infinity when consumption tends to 0.

Finally, both goods are produced if and only if the economy stands somewhere in between $p_{min}(k)$ and $p_{max}(k)$. Patterns of specialization are depicted in figure 2.

2 Preference for the present and the steady-state

Let's first look at the only possible steady-state. At this stage, no assumption about relative capital intensities is needed.

A steady-state is a pair (k^*, p^*) such that if $(k_t, p_t) = (k^*, p^*)$, then $(k_{t+1}, p_{t+1}) = (k^*, p^*)$, with respect to the dynamical system. Then, ω^* is uniquely determined by p^*, by theorem 1.1 and so are $l^C(k^*, \omega^*), k^C(\omega^*), l^I(k^*, \omega^*), k^I(\omega^*)$, as well as the consumption level c^*, by (18) and (19).

No steady-state exists in the $p < p_{min}$ region as k cannot remain constant when the economy is specialized in consumption.

At steady-state, relative price p, wage-rental ratio ω and consumption are all constant. From (21), q is also constant. From (11) and (16):

$$\beta\left(1-\delta+g'\left(k^I(\omega^*)\right)\right)=1, \tag{30}$$

which is independent of the utility function. Capital-labor ratio is obtained by the equation of zero net investment:

$$l^I(k^*,\omega^*)g(k^I(\omega^*))=\frac{k^C(\omega^*)-k^*}{k^C(\omega^*)-k^I(\omega^*)}g(k^I(\omega^*))=\delta k^* \tag{31}$$

Thus, there exists only one unique possible steady-state wage-rental ratio ω^*. If we let θ be the subjective discount rate, i.e. the positive number such that $\beta=1/(1+\theta)$, then ω^* satisfies the following fundamental relation:

$$g'(k^I(\omega^*))=\delta+\theta, \tag{32}$$

which is reminiscent of the one-sector golden rule "$f'=g+\delta$" with no technical change[12]. At the quasi-golden rule, capital per worker in the consumption sector is determined by the efficiency equation (19) while total capital stock is determined by (18) and (30):

$$g'(k^I(\omega^*))=\delta+\theta \tag{33}$$

$$\frac{f(k^C(\omega^*))}{f'(k^C(\omega^*))}-k^C(\omega^*)=\frac{g(k^I(\omega^*))}{g'(k^I(\omega^*))}-k^I(\omega^*) \tag{34}$$

$$k^*=\frac{g(k^I(\omega^*))}{\delta[k^C(\omega^*)-k^I(\omega^*)]+g(k^I(\omega^*))}k^C(\omega^*) \tag{35}$$

So the only steady-state wage-rental ratio thus depends solely on the technology of the investment sector, and not at all on the technology of the consumption sector. To see why this striking property is true, let's first look at the limit wage-rental ratio when θ goes to 0, which we dub the 'pure golden rule' wage-rental ratio ω^{GR}:

$$g'(k^I(\omega^{GR}))=\delta \tag{36}$$

[12]See Phelps (1961) and (1965). As said in the introduction of this paper, Cass (1965) presents an optimal one-sector model with discounting that exhibits the same quasi-golden rule (which he calls "modified") as we obtain here. This rule also appears in previous two-sector models including Srinivasan (1964), Uzawa (1965), Shell (1967), Haque (1970), Galor (1992) and Cremers (2005).

Clearly $\omega^{GR} > \omega^*$.

At (k^{GR}, ω^{GR}) steady-state consumption cannot be increased anymore and a supposedly-free additional unit of capital would have to be, forever and entirely, installed in the investment sector to replace its depreciated part over time. From the point of view of the social planner, it is then more profitable to set this marginal unit of capital it within the consumption sector, and to let it depreciate gradually over time[13]. As we see, the argument does not invoke the technology of the consumption sector: if capital is too costly to preserve, it can only be due to the technology of the investment sector.

At (k^*, ω^*) steady-state consumption can possibly be increased, but intertemporal benefits are less than transition costs. At ω^*, one additional unit of capital per head, if forever installed in the consumption sector, brings an intertemporal benefit of:

$$\sum_{\tau=0}^{\infty} \beta^\tau (1-\delta)^\tau f'(k^C(\omega^*))u'(c^*) = \frac{f'\left(k^C(\omega^*)\right) u'(c^*)}{\delta+\theta} \tag{37}$$

Another strategy is to first install this unit of capital in the investment sector at $\tau = 0$, letting it 'become' $(1 - \delta + g'(\omega^*))$ at $\tau = 1$; and then put this quantity in the consumption sector forever. Seen from date $\tau = 0$, intertemporal benefits are:

$$\begin{aligned} &\left(1-\delta+g'(k^I(\omega^*))\right) . \sum_{\tau=1}^{\infty} \beta^\tau (1-\delta)^\tau f'\left(k^C(\omega^*)\right) u'(c^*) \\ &= \left(1-\delta+g'(k^I(\omega^*))\right) \left[\beta \frac{f'\left(k^C(\omega^*)\right) u'(c^*)}{\delta+\theta}\right] \end{aligned} \tag{38}$$

At optimum, the social planner is indifferent between these two strategies. Equality between (37) and (38) reduces to the quasi-golden rule (32) and does not itself depend on f' nor on u'. The reason is that whatever the preferences and the production function of the consumption sector, the marginal increase in utility from installing one more unit of capital per worker in the consumption sector today $\frac{f'(k^C(\omega^*))u'(c^*)}{\delta+\theta}$ is proportional to the marginal increase in utility from installing one more unit of capital per worker in the consumption sector in the future $\beta(1-\delta+g')\frac{f'(k^C(\omega^*))u'(c^*)}{\delta+\theta}$. At the quasi-golden rule, thus, those costs and benefits balance and only the comparison between net yield $(g' - \delta)$ and subjective discount rate θ remain relevant.

3 The dynamic system

The hypothesis of no factor-intensity reversal ensures that there is a monotonic relation between p and ω, so that solving the problem through ω is virtually the same as solving the problem through p.

[13] At the margin, free capital at optimal steady-state is transitory income which it is optimal to 'consume' by setting it in the consumption sector and not replacing it when it depreciates. This is the opposite conclusion of the Friedman's rule, which sees propensity to save out of transitory income as being equal to 1.

When both goods are produced at t and at $t+1$, and in virtue of (2), (6), (11) and (16), the dynamic system evolves according to:

$$k_{t+1} - k_t = -\delta k_t + i_t \tag{39}$$

$$p_t u'(c_t) = \beta \left(1 - \delta + g'\left(k^I(\omega_{t+1})\right)\right) p_{t+1} u'(c_{t+1}) \tag{40}$$

where:

$$\begin{aligned}
i_t &= l^I(k_t, \omega_t) g(k^I(\omega_t)) = \tfrac{k^C(\omega_t) - k}{k^C(\omega_t) - k^I(\omega_t)} g(k^I(\omega_t)) \\
c_t &= l^C(k_t, \omega_t) f(k^C(\omega_t)) = \tfrac{k_t - k^I(\omega_t)}{k^C(\omega_t) - k^I(\omega_t)} f(k^C(\omega_t)) \\
c_{t+1} &= l^C(k_{t+1}, \omega_{t+1}) f(k^C(\omega_{t+1})) = l^{C\oplus}(k_t, \omega_t, \omega_{t+1}) f(k^C(\omega_{t+1})) \\
l^{C\oplus}(k_t, \omega_t, \omega_{t+1}) &= \tfrac{k_{t+1}(k_t, \omega^t) - k^I(\omega_{t+1})}{k^C(\omega_{t+1}) - k^I(\omega_{t+1})} \\
&= \tfrac{(1-\delta)k_t + \frac{k^C(\omega_t) - k_t}{k^C(\omega_t) - k^I(\omega_t)} g(k^I(\omega_t)) - k^I(\omega_{t+1})}{k^C(\omega_{t+1}) - k^I(\omega_{t+1})}
\end{aligned} \tag{41}$$

When no investment good is produced at t and when some investment good is produced at $t+1$, the system is driven by:

$$k_{t+1} - k_t = -\delta k_t \tag{42}$$

$$p_t u'(f(k_t)) = \beta \left(1 - \delta + g'\left(k^I(\omega_{t+1})\right)\right) p_{t+1} u' \left(\frac{(1-\delta)k_t - k^I(\omega_{t+1})}{k^C(\omega_{t+1}) - k^I(\omega_{t+1})} f(k^C(\omega_{t+1})) \right) \tag{43}$$

When no investment good is produced at t nor at $t+1$, in virtue of (9), the system follows:

$$k_{t+1} - k_t = -\delta k_t \tag{44}$$

$$p_t u'(f(k_t)) = \beta \left[f'\left((1-\delta)k_t\right) + (1-\delta) p_{t+1} \right] u'\left(f((1-\delta)k_{t+1})\right) \tag{45}$$

Before turning to the study of the $(k_{t+1} = k_t)$ and $(p_{t+1} = p_t)$ loci and to their stability properties, a remark is worth making. The system has two differential equations in k and p but only one initial condition (k_0). To close the optimal program, we must also impose the following natural transversality condition:

$$\beta^t q_t k_t \xrightarrow{t\to\infty} 0, \tag{46}$$

We will see that the phase diagram presents a saddle type. Condition (46) selects the unique converging path which is the stable arm of the saddle point (k^*, p^*). Remark that this stable arm is therefore nothing else than the relation between capital relative initial price p_0^* and initial capital per head k_0. Transition takes place along with this curve.

The central result we prove is the following:

Theorem 3.1. *When capital level is initially less than its steady-state optimal value, the relative price of capital in terms of the consumption good increases during transition if the consumption sector is relatively more capital-intensive, and decreases during transition if the investment sector is relatively more capital-intensive. The wage-rental ratio always increases.*

4 The optimum path: the case of a capital-intensive consumption sector

We now turn to the study of the dynamical system in the case the consumption sector is more capital-intensive than the investment sector for all wage-rental ratios. In the (k,p) plan, let $(KK) = \{(k,p)|(k_t,p_t) = (k,p) \Rightarrow k_{t+1} = k\}$ be the set of pairs (k,p) for which the capital stock remains constant from period to another, and $(PP) = \{(k,p)|(k_t,p_t) = (k,p) \Rightarrow p_{t+1} = p\}$ be the set of pairs (k,p) for which the relative price of capital remains constant from period to another[14].

4.1 The (KK) locus when the consumption sector is capital-intensive

Let's call $\widetilde{k}$ the maximum sustainable capital-labor ratio and let $\widetilde{\omega}$ denote the corresponding wage-rental ratio:

$$g(\widetilde{k}) = \delta\widetilde{k} \tag{47}$$

$$g(k^I(\widetilde{\omega})) = \delta k^I(\widetilde{\omega}) \tag{48}$$

Capital-labor ratios strictly above $\widetilde{k}$ are unsustainable for, then, even if all resources were put in the investment sector, output would not be sufficient to replace depreciated capital across time. For $k > \widetilde{k}$ therefore, the capital-labor ratio cannot remain constant. $\widetilde{k}$ is exactly sustained if the economy specializes in the production of the investment good, i.e. if $p > p_{max}$.

[14] These notations are borrowed from Galor (1992).

Define the ratio of the derivatives of technical marginal rates of substitution between the two sectors:

$$1+\lambda(\omega)=\frac{\frac{-gg''}{g'^2}(k^I(\omega))}{\frac{-ff''}{f'^2}(k^C(\omega))}>0 \tag{49}$$

From (20), we have that:

$$\frac{\partial k^C}{\partial \omega}=(1+\lambda(\omega))\frac{\partial k^I}{\partial \omega} \tag{50}$$

We first prove the following lemma:

Lemma 4.1. *When the consumption sector is relatively more capital-intensive, it holds that:*

$$\frac{\partial l^I(k,\omega)g(k^I(\omega))}{\partial \omega}>0 \tag{51}$$

$$\frac{\partial l^I(k,\omega)g(k^I(\omega))}{\partial k}<0 \tag{52}$$

Proof. Dropping the subscript ω:

$$\frac{\partial l^I}{\partial \omega}=\frac{\partial \frac{k^C-k}{k^C-k^I}}{\partial \omega}=\frac{\partial k^I}{\partial \omega}\frac{1+\lambda l^C}{k^C-k^I}.$$

As $\lambda>-1$, we have that $1+\lambda l^C>1-l^C=l^I\geq 0$. And so:

$$\frac{\partial l^I g(k^I)}{\partial \omega}=\frac{\partial k^I}{\partial \omega}\left(\underbrace{\frac{1+\lambda l^C}{k^C-k^I}g(k^I)}_{>0}+\underbrace{l^I g'(k^I)}_{>0}\right)>0.$$

On the other hand,

$$\frac{\partial l^I g(k^I)}{\partial k}=-\frac{g(k^I)}{k^C-k^I}<0,$$

which completes the proof.

$\square$

Inequality (52) reflects the validity of Rybczynski's (1955) theorem: if k increase, the output of the capital-intensive industry increases while the output of the labor-intensive industry decreases.

Inequality (51) is surprising at first sight, as one would expect an increase in the wage-rental ratio to relatively disadvantage the industry that is more labor-intensive (i.e. the investment good industry). But an increase in ω yields an increase in the relative price of the investment good p and, all things being equal, an increase in the relative profit of the investment sector that leads to an expansion in production of the investment good. The capital-labor ratio increases in both sectors as the result of substitution between inputs, but l^I increases while l^C decreases. We also show below (lemma 4.2) that and $l^C f(k^C)$ is decreasing with respect to the wage-rental ratio, reflecting a net shift of capital and labor resources from the consumption sector to the investment sector when the wage-rental ratio goes up.

For each $\omega \in (0, \widetilde{\omega})$, there exists one and only one value of k that induces zero net investment. In view of the results of lemma 4.1, the capital-labor ratio is determined like in figure 3. When ω increases, the $l^I g(k^I)$ curve shifts up, and the δk curve does not change. Because the former is a decreasing function of k, this shift induces an increase in the corresponding capital-labor ratio.

Consequently, the (KK) curve is as represented in figure 4.

The (KK) locus is stable: if we move north from a point that is on (KK) by increasing p – and so ω – while keeping k constant, real investment will tend to increase because of (51), thus driving the system toward the right, i.e. towards (KK).

4.2 The (PP) locus when the consumption sector is capital-intensive

The analysis of the (PP) curve is more delicate. We know from section 2 that (PP) and (KK) only cross at (k^*, p^*, ω^*).

Define, for each (k, ω):

$$k_\oplus = (1-\delta)k + l^I(k,\omega) g(k^I(\omega)) = (1-\delta)k + \frac{k^C(\omega) - k}{k^C(\omega) - k^I(\omega)} g(k^I(\omega)) \tag{53}$$

In the non-specialization region between the p_{min} and p_{max} curves, from (40), the relative price p remains constant from one period to the next[15] if and only if the wage-rental ratio and the capital-labor ratio verify the following equality:

$$u'(c(k,\omega)) = \beta \left(1 - \delta + g'(k^I(\omega))\right) u'(c_\oplus(k,\omega)) \tag{54}$$

where:

$$c = \frac{k - k^I(\omega)}{k^C(\omega) - k^I(\omega)} f(k^C(\omega)) \tag{55}$$

$$c_\oplus = \frac{k_\oplus - k^I(\omega)}{k^C(\omega) - k^I(\omega)} f(k^C(\omega)) \tag{56}$$

[15] Once again, recall that p remains constant if and only if ω remains constant.

When the capital-labor ratio stands initially at k and that the wage-rental ratio stays at ω at current and next period, c and $c_\oplus$ therefore respectively denote current and next period's consumption.

In the non-specialization region, (54) determines a unique wage-rental ratio $\omega^{PP}(k)$ that is consistent with stability of relative prices from one period to the next.

Next step is to show that (PP) is increasing in the (k, p) plan. The local result always holds with no further assumption. The semi-global conclusion will necessitate two other assumptions. We therefore separate the two.

Let's first enunciate a lemma that always holds:

Lemma 4.2. *For all $\omega > 0$,*

$$\frac{\partial l^C}{\partial \omega} < 0 \tag{57}$$

$$\frac{\partial c}{\partial \omega} < 0 \tag{58}$$

Proof. See appendix. □

Proposition 4.1. *When the consumption sector is relatively capital-intensive, function is $\omega^{PP}(k)$ locally strictly increasing around quasi-golden rule capital-labor ratio k^*.*

Proof.

Lemma 4.3.

$$\left.\left(\frac{\partial l^{C\oplus}}{\partial \omega} - \frac{\partial l^C}{\partial \omega}\right)\right|_{\omega=\omega^*} > 0 \tag{59}$$

$$\left.\frac{\partial\left(\frac{u'(c)}{u'(c_\oplus)}\right)}{\partial \omega}\right|_{\omega=\omega^*} > 0 \tag{60}$$

$$\left.\frac{\partial\left(\frac{u'(c)}{u'(c_\oplus)}\right)}{\partial k}\right|_{\omega=\omega^*} < 0 \tag{61}$$

Proof. See appendix for the proof of the lemma. □

From (60) and (61), around balanced capital-labor ratio k^*, the ω^{PP} wage-rental ratio is determined like in figure 5 which proves graphically – but the algebra is straightforward – that (PP) is locally strictly increasing at quasi-golden rule steady-state[16]. □

[16]Remark that, for the conclusion to hold, it is sufficient that $\left.\frac{\partial\left(\frac{u'(c)}{u'(c_\oplus)}\right)}{\partial \omega}\right|_{\omega=\omega^*} > \beta g''(k^I)\frac{\partial k^I}{\partial \omega}$.

We now show that (PP) crosses (KK) from above:

Proposition 4.2. *Around steady-state capital-labor ratio k^*, $\omega^{PP}(k) > \omega^{KK}(k)$ for $k < k^*$ and $\omega^{PP}(k) < \omega^{KK}(k)$ for $k > k^*$.*

Proof. Because (PP) is locally strictly upward-sloping, then $\omega^{PP}(k) < \omega^*$ for $k < k^*$. Consequently, for $k < k^*$ we have that $\beta\left(1 - \delta + g'(k^I(\omega^{PP}))\right) > 1$ which, in virtue of (54), implies that $c < c_\oplus$ on the (PP) locus. From (53), (55) and (56), this implies that the (PP) locus lies in the region where $l^I g(k^I) > \delta k$ when $k < k^*$. Because the (KK) branch is stable, this region is located above (KK).

□

We now show that the result that (PP) is locally upward-sloping can be generalized to the $(0, k^*)$ interval – which is the one of interest if one focuses on the case of an initially under-capitalized economy – at the cost of two additional assumptions:

Proposition 4.3. *Suppose that absolute risk-aversion coefficient $\left(-\frac{u''(c)}{u'(c)}\right)$ is decreasing and that $\forall \omega \in (0, \omega^*)$, $\lambda(\omega) \in (-1, 1)$. Then function $\omega^{PP}(k)$ is strictly increasing for $k < k^*$, and so the (PP) curve is increasing in the (k, p) plan for $k < k^*$.*

Proof. These two additional assumptionFFirst, remark that from proposition (4.2), we know that for $k < k^*$, (PP) lies above (KK). Au s (KK) is stable, when $(k, p) \in (PP)$, k and p necessarily verify that $l^I(k, \omega) g(k^I(\omega)) > \delta k$ where $\omega = \omega(p)$ like implicitly defined by theorem 1.1. And so, from (53) and (56), we have that $l^{C\oplus} > l^C$ and $c_\oplus > c$ on the (PP) curve for $k < k^*$.

Lemma 4.4. *When absolute risk-aversion $-\frac{u''(c)}{u'(c)}$ is decreasing and that $\forall \omega \in (0, \omega^*)$, $\lambda(\omega) \in (-1, 1)$:*

$$\left.\frac{\partial (l^{C\oplus} - l^C)}{\partial \omega}\right|_{\omega \leq \omega^*} > 0 \tag{62}$$

$$\left.\frac{\partial \frac{u'(c)}{u'(c_\oplus)}}{\partial \omega}\right|_{\omega \leq \omega^*} > 0 \tag{63}$$

$$\left.\frac{\partial \frac{u'(c)}{u'(c_\oplus)}}{\partial k}\right|_{\omega \leq \omega^*} < 0 \tag{64}$$

Proof. See appendix for the proof of the lemma. □

Thereby, when $\omega \leq \omega^*$, the situation is still like depicted in figure 5 and the (PP) curve is increasing at the left of steady-state. □

Final step to prove the saddle property of the (k, p) system is to show that the (PP) locus is unstable:

Proposition 4.4. *When the consumption sector is relatively capital-intensive, the (PP) locus is unstable.*

Proof. In the (p_{min}, p_{max}) region, and from equation (40), p changes through time according to:

$$\frac{p_{t+1}}{p_t} = \frac{u'(c_t)}{\beta\left(1-\delta+g'(k^I(\omega_{t+1}))\right)u'(c_{t+1})}, \tag{65}$$

Suppose, for example, that the system initially stands at steady-state and suddenly moves right at t, so that k increases while ω stays at ω^*. From (61), we conclude that $\frac{u'(c_t)}{\beta(1-\delta+g'(k^I(\omega^*)))u'(c_{t+1})}$ shall tend to decrease. From (65), $\frac{p_{t+1}}{p_t}$ will also tend to decrease from a value of 1, and so p_{t+1} will become less than p_t. As (PP) is upward-sloping, this proves that the (PP) locus is locally unstable around p^*.

□

4.3 The local saddle

In view of the results above, we are now able to represent the local phase diagram in figure 6. (PP) crosses (KK) from above at the quasi-golden rule steady-state and both curves are locally increasing. (KK) is stable and (PP) is unstable, so that the steady-state is a saddle point. The stable arm is represented by the arrowed curve. Any path reaching the north-east or the south-west quadrants diverges. We see graphically that k and p must necessarily move in same direction along the only stable path.

So locally, the relative price of the investment good and the wage-rental ratio *increase* during transition if initial capital-labor ratio is less than k^*. The opposite happens in the case the economy starts with 'too-much' capital. Notice that this result does not necessitate any assumption about risk-aversion nor about the coefficient λ.

4.4 The optimal transition path

If we take the two additional assumptions that absolute risk aversion $-\frac{u''(c)}{u'(c)}$ is decreasing with respect to c and that $\lambda(\omega)$ always stands at less than 1 for $\omega \leq \omega^*$, then the (PP) curve is upward-sloping in the $(0, k^*)$ interval. Consequently, the stable arm of the saddle is globally increasing for $k \leq k^*$. This is summarized by figure 7. An economy starting from an initial capital-labor of $k_0 < k^*$ sees relative price of investment initially set at p_0[17] gradually increase during transition towards $p^* > p_0$.

5 The optimum path: the case of a capital-intensive investment sector

The formal identities we have put for the case where the consumption sector is relatively more capital-intensive are still valid, but often conclusions will be opposite. Still, it is straightforward to adapt the results of the previous section to the case where $k^I(\omega) > k^C(\omega)$ for all $\omega > 0$.

[17]Recall that determinacy of the initial relative price p_0 and of initial wage-rental ratio ω_0 is a consequence of the transversality condition (46) imposed to the dynamical system.

From theorem 1.1, there exists now a *negative* correspondence between p and ω: an increase in ω increases the relative costs of the consumption industry (which is now labor-intensive) which in turn make consumption goods relatively more expensive.

5.1 The (KK) locus when the investment sector is capital-intensive

Equation of the (KK) locus in the non-specialization region of the (k,p) plan between p_{min} and p_{max} is now:

$$\frac{k - k^C(\omega)}{k^I(\omega) - k^C(\omega)} g(k^I(\omega)) = \delta k, \tag{66}$$

which determines, for all $k < \widetilde{k}$ one unique wage-rental ratio $\omega^{KK}(k)$ consistent with zero net investment, where $\widetilde{k}$ is the maximum sustainable capital-labor ratio defined in (47).

Let's first prove the following lemma:

Lemma 5.1. *When the investment sector is relatively more capital-intensive:*

$$\frac{\partial l^I(k,\omega)}{\partial \omega} < 0 \tag{67}$$

$$\frac{\partial l^I(k,\omega) g(k^I(\omega))}{\partial \omega} < 0 \tag{68}$$

And on the (KK) curve, it holds that:

$$\left.\frac{\partial l^I(k,\omega) g(k^I(\omega))}{\partial k}\right|_{l^I g(k^I)=\delta k} > \delta (> 0) \tag{69}$$

Proof.

- $\frac{\partial l^I}{\partial \omega} = -\frac{\partial k^I}{\partial \omega} \frac{1+\lambda l^C}{k^I - k^C} < 0$.
- $\frac{\partial l^I g(k^I)}{\partial \omega} = \frac{\partial k^I}{\partial \omega} \left(-(1+\lambda l^C) \frac{g(k^I)}{k^I - k^C} + l^I g'(k^I) \right)$

  But $\lambda > -1 \Rightarrow -(1+\lambda l^C) < -l^I$ and $\frac{g(k^I)}{k^I - k^C} > \frac{g(k^I)}{k^I} > g'(k^I)$, which permits to conclude that $\frac{\partial l^I g(k^I)}{\partial \omega} < 0$.
- $\frac{\partial l^I g(k^I)}{\partial k} = \frac{g(k^I)}{k^I - k^C} = \frac{l^I g(k^I)}{k - k^C}$ and so:

  $$\left.\frac{\partial l^I(k,\omega) g(k^I(\omega))}{\partial k}\right|_{l^I g(k^I)=\delta k} = \frac{\delta k}{k - k^C} = \frac{\delta}{1 - \frac{k^C}{k}}$$

  which is strictly greater than δ because $k > k^C$ when the investment sector is relatively more capital-intensive[18].

$\square$

[18] Remark this is a strong version of the Rybczynski (1955) theorem.

Therefore, for each $k < \widetilde{k}$, $\omega^{KK}(k)$ is determined like in figure 8, which demonstrates graphically that ω^{KK} is increasing with respect to k, meaning that the (KK) locus is decreasing in the (k,p) plan (because of theorem 1.1) for $k \in (0, \widetilde{k})$. Figure 9 represents the (KK) locus when the investment sector is relatively more capital-intensive.

The (KK) locus is now unstable: from (69) we know that $\frac{\partial(l^I g(k^I) - \delta k)}{\partial k} > 0$ on the portion of (KK) that lies in the non-specialization region. Therefore, if we move right from a point on the decreasing part of (KK), this increase in k will induce an increase in net investment, making the capital stock increase and the economy shift further to the right.

5.2 The (PP) locus when the investment sector is capital-intensive

The general shape of the (PP) locus is undefined, even for $k < k^*$ and with the assumptions made in section 4 when consumption was supposed to be relatively capital-intensive. But we show that (PP) is locally downward-sloping and it crosses (KK) from below. In the case of an initially under-capitalized economy, we show graphically that the optimum path goes along the (KK) curve from above, and, consequently, we suggest that the form of (PP) shape is of secondary importance. As the construct of the (global) (KK) locus did not require any assumption, so will the shape of the optimal path for an economy starting from arbitrarily low.

The equation of the (PP) curve is still the same:

$$u'(c(k,\omega)) = \beta \left(1 - \delta + g'(k^I(\omega))\right) u'(c_{\oplus}(k,\omega)), \tag{70}$$

where, for each (k,ω):

$$k_{\oplus} = (1-\delta)k + l^I(k,\omega)g(k^I(\omega)) = (1-\delta)k + \frac{k - k^C(\omega)}{k^I(\omega) - k^C(\omega)} g(k^I(\omega)) \tag{71}$$

$$c = \frac{k^I(\omega) - k}{k^I(\omega) - k^C(\omega)} f(k^C(\omega)) \tag{72}$$

$$c_{\oplus} = \frac{k^I(\omega) - k_{\oplus}}{k^I(\omega) - k^C(\omega)} f(k^C(\omega)) \tag{73}$$

Those expressions are the same as in the preceding section and are simply rewritten so that all numerators and denominators are positive.

The whole implicitly defines, in the neighborhood of k^* and in the $(0, k^*)$ a unique $\omega^{PP}(k) > 0$ that satisfies (70).

Consumption is now always increasing in the wage-rental ratio, as:

$$\frac{\partial l^C}{\partial \omega} = \frac{\partial k^I}{\partial \omega}\frac{1+\lambda l^C}{k^I - k^C} > 0 \tag{74}$$

$$\frac{\partial l^C f(k^C)}{\partial \omega} = \frac{\partial l^C}{\partial \omega} f(k^C) + l^C \frac{\partial k^C}{\partial \omega} f'(k^C) > 0 \tag{75}$$

By the following proposition, we conclude that the (PP) curve is now downward-sloping in the (k,p) plan:

Proposition 5.1. *When the investment sector is relatively capital-intensive, function is $\omega^{PP}(k)$ locally strictly increasing around quasi-golden rule capital-labor ratio k^*.*

Proof. Inequalities of lemma 4.3 are not modified when the investment sector is supposed to be the capital-intensive sector:

Lemma 5.2.

$$\left(\frac{\partial l^{C\oplus}}{\partial \omega} - \frac{\partial l^C}{\partial \omega} \right)\Bigg|_{\omega=\omega^*} > 0 \tag{76}$$

$$\frac{\partial \left(\frac{u'(c)}{u'(c_\oplus)} \right)}{\partial \omega}\Bigg|_{\omega=\omega^*} > 0 \tag{77}$$

$$\frac{\partial \left(\frac{u'(c)}{u'(c_\oplus)} \right)}{\partial k}\Bigg|_{\omega=\omega^*} < 0 \tag{78}$$

Proof. See appendix for the proof of the lemma. □

Hence, in the neighborhood of k^*, $\omega^{PP}(k)$ is – as in section 4 – determined like in figure 5 when the investment sector is capital-intensive, and $\omega^{PP}(k)$ is locally increasing around k^*.

□

As p and ω follow a negative relationship by theorem 1.1, the (PP) curve is locally downward-sloping around steady-state in the (k,p) space when the investment sector is capital-intensive.

(PP) crosses (KK) from below in the (k,p) plan when the investment sector is capital-intensive:

Proposition 5.2. *Around steady-state capital-labor ratio k^*, $\omega^{PP}(k) > \omega^{KK}(k)$ for $k < k^*$ and $\omega^{PP}(k) < \omega^{KK}(k)$ for $k > k^*$.*

Proof. Because $\omega^{PP}(k)$ is locally increasing around k^*, $\omega^{PP}(k) < \omega^*$ for $k < k^*$. Consequently, for $k < k^*$ we have that $\beta\left(1-\delta+g'(k^I(\omega^{PP}))\right) > 1$ which, in virtue of (70), implies that $c < c_\oplus$ on the (PP) locus. From (71), (72) and (73), this implies that the (PP) locus lies in the region where $l^I g(k^I) < \delta k$ when $k < k^*$. Because the (KK) locus is unstable, this region is located below (KK).

□

The (PP) branch is now stable. p changes through time according to:

$$\frac{p_{t+1}}{p_t} = \frac{u'(c_t)}{\beta\left(1-\delta+g'(k^I(\omega_{t+1}))\right) u'(c_{t+1})} \tag{79}$$

Suppose that the economy initially stands at the quasi-golden rule steady-state, and suddenly experiences a shift to the right at t, so that k increases while ω stays at ω^*. Then, from equation (78), $\frac{u'(c_t)}{\beta(1-\delta+g'(k^I(\omega^*)))u'(c_{t+1})}$ will tend to decrease. From (79), $\frac{p_{t+1}}{p_t}$ will be less than one which proves that p will tend to decrease. As (PP) is downward-sloping, this proves that the (PP) locus is locally unstable around p^*.

5.3 The local saddle

The local phase diagram is represented in figure 10. (KK) and (PP) are both decreasing in the around (k^*, p^*) and (KK) is initially above (PP). As (KK) and (PP) are respectively stable and unstable, the system displays a local saddle which stable arm is sketched with arrows. Locally, along the optimal path, p (resp. ω) and k move in opposite (resp. same) direction.

5.4 The optimal transition path

As figure 10, the optimal path tends to follow the (KK) curve, while it tended to follow the (PP) locus when the consumption sector was supposed to be relatively capital-intensive. Here, (PP) lies below (KK) for $k < k^*$, while the optimal path is above (KK). Hence, the exact shape of the (PP) curve when $k < k^*$ is of secondary interest. As it does not require further assumptions to prove that the (KK) locus is downward-sloping in the (k, p) plan when $k < k^*$, we conclude that the optimal path of accumulation for an economy starting at $k_0 < k^*$ is also monotonically downward-sloping and lies above (KK). This path is represented in figure 11.

6 Some comparative dynamics: the effect of a negative shock on the discount rate

An interesting exercise is to analyze the dynamic effect of a decrease of the rate of preference for the present θ. When the agents are more patient, the new steady-state is characterized by a higher wage-rental ratio ω^* (equation (32)), higher capital intensities in both sectors (equations (19) and (20)) and a higher aggregate capital-labor ratio (equation (35)).

Suppose that the economy initially stands at the (k^*, p^*) steady-state consistent with a rate of preference for the present $\theta > 0$, and suppose that this parameter θ unexpectedly falls to zero at date t and forever. The economy's new steady-state corresponds to the two-sector version of the pure Phelpsian golden rule $g' = \delta$ where steady-state consumption is maximized (see equation (36)).

Equation of (KK) is independent of θ and so the (KK) locus remains unchanged. We discussed above that the (PP) locus is determined by a graph like that of figure 5. When β increases, $\frac{u'(c)}{u'(c_\oplus)}$ remains unchanged and $\beta(1-\delta+g') = \frac{1}{1+\theta}(1-\delta+g')$ goes up. As the situation depicted in figure 5 holds regardless of the capital-intensive sector, $\omega^{PP}(k)$ increases for all k in both cases. In virtue of theorem 1.1, (PP) shifts up in the case the consumption sector is relatively capital-intensive and it shifts down if the investment sector is relatively capital-intensive. We see in figure 12 that the optimal path shifts up in both cases.

What happens after this unexpected shock? At t, the capital stock is fixed at k^* corresponding to pre-shock steady-state. When the shock hits, the relative price of capital adjusts so as to put the economy immediately on its (new) optimal path to golden rule steady-state. Because the optimal path shifts up regardless of the capital-intensity assumption, we can conclude that p jumps up in any case: the fall of the rate of impatience to zero induces an immediate positive jump in the relative price of capital in terms of consumption good. As the discount rate decreases, the price of capital is instantly positively revalued.

In a second phase, the economy gradually converges to golden rule steady-state. As capital accumulates, labor becomes relatively-scarce in relation to capital, and so the wage-rental ratio increases during transition. In the case the consumption sector is more capital-intensive, relative price of capital thus tends to increase during this phase. In the case the investment sector is relatively capital-intensive, it gradually decreases toward a value that stands at less than its pre-shock steady-state level.

7 Concluding remarks

In the two-sector Ramsey framework presented, any economy that starts from a capital-ratio k_0 that is less than the balanced ratio k^* monotonically converges to steady-state (k^*, p^*).

Optimal transition always takes place with increases in relative wages, independently of any capital-intensity assumption. Because supply of one factor of production (labor) remains constant while supply of the other one (capital) increases, relative remuneration must must follow the opposite pattern. The flow's approach conclusion is that relative rental decreases as capital is accumulated. Figure 13 represents the local phase diagrams in the (k, ω) space which highlights the fact that, while the dynamic system is *qualitatively* different in the two cases, the optimal path of accumulation always goes with an increase in relative wages.

We would tend to think that, from a stock perspective, capital would become cheaper as it becomes more abundant along with the transition. Perhaps surprisingly, it happens that this conclusion necessarily rests on a capital-intensity assumption. If the consumption sector is relatively capital-intensive, then capital gains are natural during transition; but if it is the investment sector that is relatively capital-intensive, transition takes place with a smooth decrease in relative capital price. This is because of the Stolper-Samuelson-Uzawa effect: as capital become more abundant, its (macroeconomic) reward must necessarily goes down, which relatively favors the costs of the capital-intensive sector.

The model presented is not, in absence of technical progress, able to give any insight in the growth phenomenon. But the centrality of the capital sector in the expression of the (quasi-)golden rule signals that technical progress will likely induce different effects on consumption and GDP, but also on wages, relative prices, asset prices, etc. according to the sector which sees its efficiency increase. Gort, Greenwood and Rupert (1999) and Greenwood, Hercowitz and Krusell (1997) show that post-war US data seems to show that investment-specific technical change is responsible for an overwhelming share of growth. This is scope for future theoretical research.

8 Appendix

8.1 The critical relative price ratio and the production frontier

Figure 14 represents, for a certain capital endowment per unit of labor, the production possibility frontier. The hypothesis of diminishing marginal returns implies that this curve has the represented concave form. $-p_{min}(k)$ is the slope of this curve at the point $(c = 0, i = g(k))$ while $-p_{max}(k)$ is the slope of the same curve at the point where $(c = f(k), i = 0)$. When k increases, the frontier moves northeast. In the case where the consumption sector is more capital-intensive, p_{min} and p_{max} tend to increase too, while the contrary happens when the investment sector is more capital intensive. These two cases are depicted in figure 15.

8.2 Proof of lemma (4.2)

- $\frac{\partial l^C}{\partial \omega} = \frac{\partial \frac{k-k^I}{k^C-k^I}}{\partial \omega} = \frac{-\frac{\partial k^I}{\partial \omega}(k^C-k^I)-(k-k^I)(\partial k^C/\partial\omega - \partial k^I/\partial\omega)}{(k^C-k^I)^2}$. Consequently, from (50):

$$\frac{\partial l^C}{\partial \omega} = -\frac{\partial k^I}{\partial \omega}\frac{1+\lambda l^C}{k^C-k^I}, \tag{80}$$

  which is negative because $\lambda > -1 \Rightarrow 1 + \lambda l^C > 1 - l^C = l^I \geq 0$ and $k^C - k^I > 0$.

- $\frac{\partial c}{\partial \omega} = \frac{\partial l^C f(k^C)}{\partial \omega} = \frac{\partial k^I}{\partial \omega}\left((1+\lambda)l^C f'(k^C) - (1+\lambda l^C)\frac{f(k^C)}{k^C-k^I}\right)$. Because $(1+\lambda)l^C \leq 1 + \lambda l^C$, we have that:

$$\frac{\partial c}{\partial \omega} \leq \frac{\partial k^I}{\partial \omega}(1+\lambda)l^C\left(\underbrace{f'(k^C) - \frac{f(k^C)}{k^C-k^I}}_{<0}\right) < 0.$$

8.3 Proof of lemma (4.3)

- $\frac{\partial (l^{C\oplus} - l^C)}{\partial \omega} = \frac{\partial}{\partial \omega}\frac{l^I g(k^I) - \delta k}{k^C - k^I}$. And so, for all $\omega > 0$,

$$\frac{\partial (l^{C\oplus} - l^C)}{\partial \omega} = \frac{\partial k^I}{\partial \omega}\frac{1}{k^C-k^I}\left(\frac{1+\lambda l^C}{k^C-k^I}g(k^I) + l^I g'(k^I) - \lambda\frac{l^I g(k^I) - \delta k}{k^C-k^I}\right) \tag{81}$$

  At $\omega = \omega^*$, the last term is zero. More over, $\lambda > -1 \Rightarrow 1 + \lambda l^C > 1 - l^C = l^I \geq 0$ and so:

$$\left.\frac{\partial (l^{C\oplus} - l^C)}{\partial \omega}\right|_{\omega=\omega^*} = \frac{\partial k^I}{\partial \omega}\frac{1}{k^C-k^I}\left(\frac{1+\lambda l^C}{k^C-k^I}g(k^I) + l^I g'(k^I)\right) > 0.$$

- $\frac{\partial \ln\left(\frac{u'(c)}{u'(c_\oplus)}\right)}{\partial \omega} = \frac{\partial c_\oplus}{\partial \omega}\left(-\frac{u''(c_\oplus)}{u'(c_\oplus)}\right) - \frac{\partial c}{\partial \omega}\left(-\frac{u''(c)}{u'(c)}\right)$. But at $\omega = \omega^*$, $l^{C\oplus} = l^C$, $c_\oplus = c$, and:

$$
\begin{aligned}
\left.\frac{\partial \ln\left(\frac{u'(c)}{u'(c_\oplus)}\right)}{\partial \omega}\right|_{\omega=\omega^*} &= \left(\frac{\partial c_\oplus}{\partial \omega} - \frac{\partial c}{\partial \omega}\right)\left(-\frac{u''(c)}{u'(c)}\right) \\
&= \left(\frac{\partial (l^{C\oplus}-l^C)f(k^C)}{\partial \omega}\right)\left(-\frac{u''(c)}{u'(c)}\right) \\
&= \left[\left(\underbrace{\frac{\partial l^{C\oplus}}{\partial \omega} - \frac{\partial l^C}{\partial \omega}}_{>0}\right) f(k^C) + (\underbrace{l^{C\oplus} - l^C}_{=0})\frac{\partial k^C}{\partial \omega} f'(k^C)\right]\underbrace{\left(-\frac{u''(c)}{u'(c)}\right)}_{>0} \\
\left.\frac{\partial \ln\left(\frac{u'(c)}{u'(c_\oplus)}\right)}{\partial \omega}\right|_{\omega=\omega^*} &> 0.
\end{aligned}
$$

- $\frac{\partial \ln\left(\frac{u'(c)}{u'(c_\oplus)}\right)}{\partial k} = \frac{\partial c_\oplus}{\partial k}\left(-\frac{u''(c_\oplus)}{u'(c_\oplus)}\right) - \frac{\partial c}{\partial k}\left(-\frac{u''(c)}{u'(c)}\right)$. But $\frac{\partial c}{\partial k} = \frac{1}{k^C-k^I}f'(k^C) > 0$ and $\frac{\partial c_\oplus}{\partial k} = \frac{1-\delta-\frac{g(k^I)}{k^C-k^I}}{k^C-k^I}f'(k^C) < \frac{\partial c}{\partial k}$. But at $\omega = \omega^*$, $c_\oplus = c$ and so:

$$
\left.\frac{\partial \ln\left(\frac{u'(c)}{u'(c_\oplus)}\right)}{\partial k}\right|_{\omega=\omega^*} = \left(\underbrace{\frac{\partial c_\oplus}{\partial k} - \frac{\partial c}{\partial k}}_{<0}\right)\left(\underbrace{-\frac{u''(c)}{u'(c)}}_{>0}\right) < 0.
$$

8.4 Proof of lemma (4.4)

- (81) holds for all $\omega > 0$, and so it holds for $\omega \leq \omega^*$. When $\omega \leq \omega^*$, $l^I g(k^I) - \delta k > 0$. And so, if $\lambda \in (-1, 0)$, then $-\lambda \frac{l^I g(k^I) - \delta k}{k^C - k^I} > 0$. and so $\frac{\partial (l^{C\oplus} - l^C)}{\partial \omega} > 0$. If $\lambda \in (0, 1)$, then (and also because $1 + \lambda l^C > l^I$):

$$
\begin{aligned}
\frac{\partial (l^{C\oplus}-l^C)}{\partial \omega} &> \frac{\partial k^I}{\partial \omega}\frac{1}{k^C-k^I}\left(\frac{l^I}{k^C-k^I}g(k^I) + l^I g'(k^I) - \lambda \frac{l^I g(k^I)-\delta k}{k^C-k^I}\right) \\
&= \frac{\partial k^I}{\partial \omega}\frac{1}{k^C-k^I}\left((1-\lambda)\frac{l^I g(k^I)-\delta k}{k^C-k^I} + l^I g'(k^I) + \lambda \frac{\delta k}{k^C-k^I}\right) \\
&> 0.
\end{aligned}
$$

- $\frac{\partial (l^{C\oplus}-l^C)}{\partial \omega} > 0$ implies that

$$
\frac{\partial (c_\oplus - c)}{\partial \omega} = \frac{\partial k^I}{\partial \omega}\left(\frac{\partial (l^{C\oplus} - l^C)}{\partial \omega} f(k^C) + (1+\lambda) f'(k^C)\right) > 0 \tag{82}
$$

When $\omega < \omega^*$, $c_\oplus > c$ along (PP) and, by the assumption made, we have that $-\frac{u''(c_\oplus)}{u'(c_\oplus)} < -\frac{u''(c)}{u'(c)}$. If $\frac{\partial c_\oplus}{\partial \omega} > 0$, then we immediately have that[19]:

$$
\frac{\partial \ln\left(\frac{u'(c)}{u'(c_\oplus)}\right)}{\partial \omega} = \underbrace{\frac{\partial c_\oplus}{\partial \omega}}_{>0}\left(-\frac{u''(c_\oplus)}{u'(c_\oplus)}\right) - \underbrace{\frac{\partial c}{\partial \omega}}_{<0}\left(-\frac{u''(c)}{u'(c)}\right) > 0.
$$

[19] Remark that, if $\frac{\partial c_\oplus}{\partial \omega} > 0$, the proof does not require the assumption that absolute risk-aversion is decreasing. In this case, an increase in the wage-rental ratio today leads to a decrease in consumption and an increase in investment today. But this increase in investment increases tomorrow's capital stock. And as $\frac{\partial c}{\partial k} > 0$, $c_\oplus$ tends to decrease less consecutively to an increase in ω than c. If $\frac{\partial c_\oplus}{\partial \omega} > 0$, it means that this intertemporal effect is strong enough to make $c_\oplus$ increase consecutively to an increase of the wage-rental ratio. A sufficient condition for that is that $\frac{\partial l^{C\oplus}}{\partial \omega} > 0$.

If $0 > \frac{\partial c_\oplus}{\partial \omega} > \frac{\partial c}{\partial \omega}$, then:

$$\left(0 < -\frac{\partial c_\oplus}{\partial \omega} < -\frac{\partial c}{\partial \omega} \text{ and } 0 < -\frac{u''(c_\oplus)}{u'(c_\oplus)} < -\frac{u''(c)}{u'(c)}\right) \Rightarrow -\frac{\partial c_\oplus}{\partial \omega}\left(-\frac{u''(c_\oplus)}{u'(c_\oplus)}\right) < -\frac{\partial c}{\partial \omega}\left(-\frac{u''(c)}{u'(c)}\right)$$
$$\Rightarrow \frac{\partial \ln\left(\frac{u'(c)}{u'(c_\oplus)}\right)}{\partial \omega} > 0.$$

- Similarly, when $c_\oplus \geq c$ – and consequently $-\frac{u''(c_\oplus)}{u'(c_\oplus)} < -\frac{u''(c)}{u'(c)}$ – we have that:

$$\begin{aligned}\frac{\partial \ln\left(\frac{u'(c)}{u'(c_\oplus)}\right)}{\partial k} &= \frac{\partial c_\oplus}{\partial k}\left(-\frac{u''(c_\oplus)}{u'(c_\oplus)}\right) - \frac{\partial c}{\partial k}\left(-\frac{u''(c)}{u'(c)}\right) \\ &\leq \left(\underbrace{\frac{\partial c_\oplus}{\partial k} - \frac{\partial c}{\partial k}}_{<0}\right)\left(-\frac{u''(c)}{u'(c)}\right) \\ &< 0.\end{aligned}$$

8.5 Proof of lemma (5.2)

- Equation (81) still formally holds. Let's rewrite it as:

$$\frac{\partial(l^{C\oplus} - l^C)}{\partial \omega} = \frac{\partial k^I}{\partial \omega}\frac{1}{k^I - k^C}\left(\frac{1 + \lambda l^C}{k^I - k^C} g(k^I) - l^I g'(k^I) - \lambda \frac{l^I g(k^I) - \delta k}{k^I - k^C}\right) \tag{83}$$

At $\omega = \omega^*$, we have that:

$$\frac{\partial(l^{C\oplus} - l^C)}{\partial \omega} = \frac{\partial k^I}{\partial \omega}\frac{1}{k^I - k^C}\left(\frac{1 + \lambda l^C}{k^I - k^C} g(k^I) - l^I g'(k^I)\right) > \frac{\partial k^I}{\partial \omega}\frac{1}{k^I - k^C}\left(\frac{l^I}{k^I - k^C} g(k^I) - l^I g'(k^I)\right),$$

which is positive since $\frac{g(k^I)}{k^I - k^C} > \frac{g(k^I)}{k^I} > g'(k^I)$.

- On the (PP) curve at $\omega = \omega^*$, $c = c_\oplus$ and:

$$\begin{aligned}\left.\frac{\partial \ln\left(\frac{u'(c)}{u'(c_\oplus)}\right)}{\partial \omega}\right|_{\omega=\omega^*} &= \left(\frac{\partial c_\oplus}{\partial \omega} - \frac{\partial c}{\partial \omega}\right)\left(-\frac{u''(c)}{u'(c)}\right) \\ &= \left[\left(\underbrace{\frac{\partial l^{C\oplus}}{\partial \omega} - \frac{\partial l^C}{\partial \omega}}_{>0}\right) f(k^C) + (\underbrace{l^{C\oplus} - l^C}_{=0})\frac{\partial k^C}{\partial \omega} f'(k^C)\right]\underbrace{\left(-\frac{u''(c)}{u'(c)}\right)}_{>0} \\ \left.\frac{\partial \ln\left(\frac{u'(c)}{u'(c_\oplus)}\right)}{\partial \omega}\right|_{\omega=\omega^*} &> 0.\end{aligned}$$

- In the case the investment sector is relatively capital-intensive, we have that:

$$\frac{\partial c}{\partial k} = \frac{-1}{k^I - k^C} f(k^C) < 0$$

$$\frac{\partial c_\oplus}{\partial k} = \frac{-1 - \delta - \frac{g(k^I)}{k^I - k^C}}{k^I - k^C} f(k^C) < \frac{\partial c}{\partial k}$$

Consequently:

$$
\begin{aligned}
\left.\frac{\partial \ln\left(\frac{u'(c)}{u'(c_\oplus)}\right)}{\partial k}\right|_{\omega=\omega^*} &= \frac{\partial c_\oplus}{\partial k}\left(-\frac{u''(c_\oplus)}{u'(c_\oplus)}\right) - \frac{\partial c}{\partial k}\left(-\frac{u''(c)}{u'(c)}\right) \\
&= \left(\underbrace{\frac{\partial c_\oplus}{\partial k} - \frac{\partial c}{\partial k}}_{<0}\right)\left(-\frac{u''(c)}{u'(c)}\right) \\
&< 0.
\end{aligned}
$$

Figures

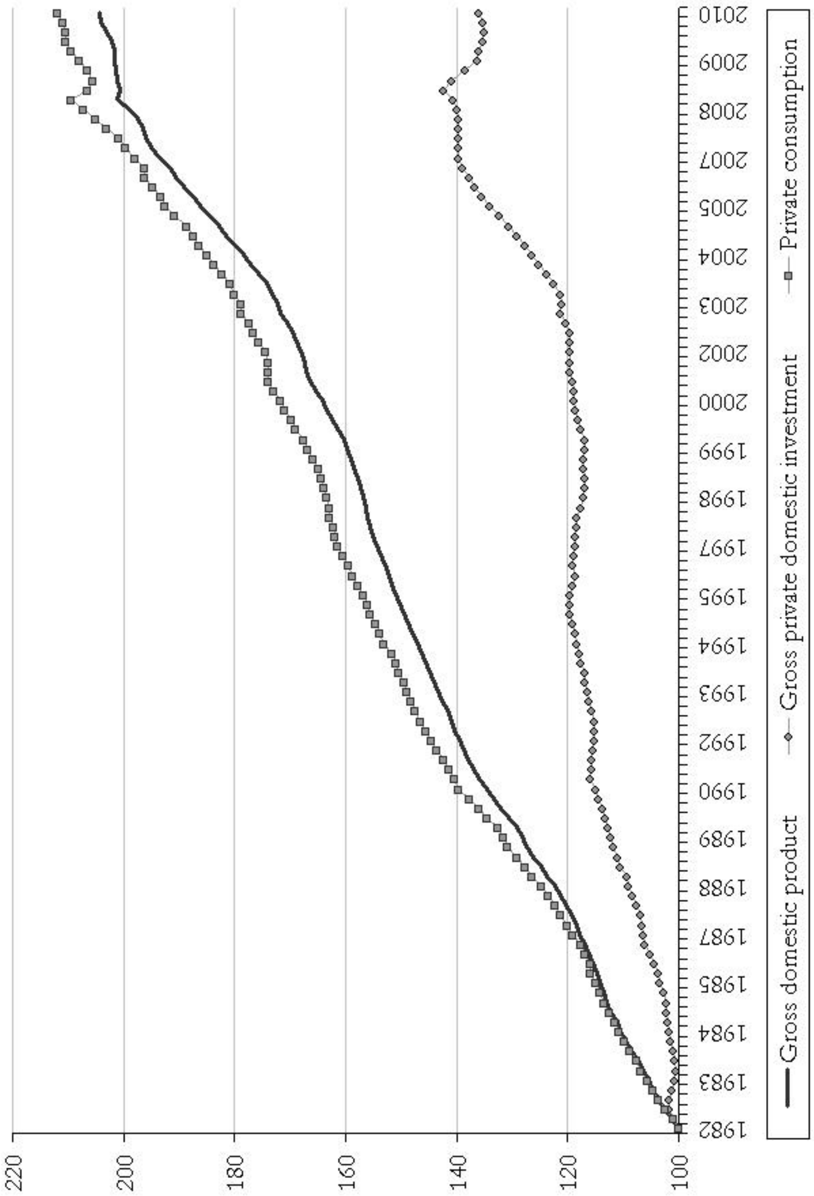

Figure 1: Price deflators for GDP, private consumption and private fixed investment, USA, 1982–2010. Quarterly, seasonally-adjusted data (source: Bureau of Economic Analysis, NIPA database (www.bea.gov). 1982-I = 100).

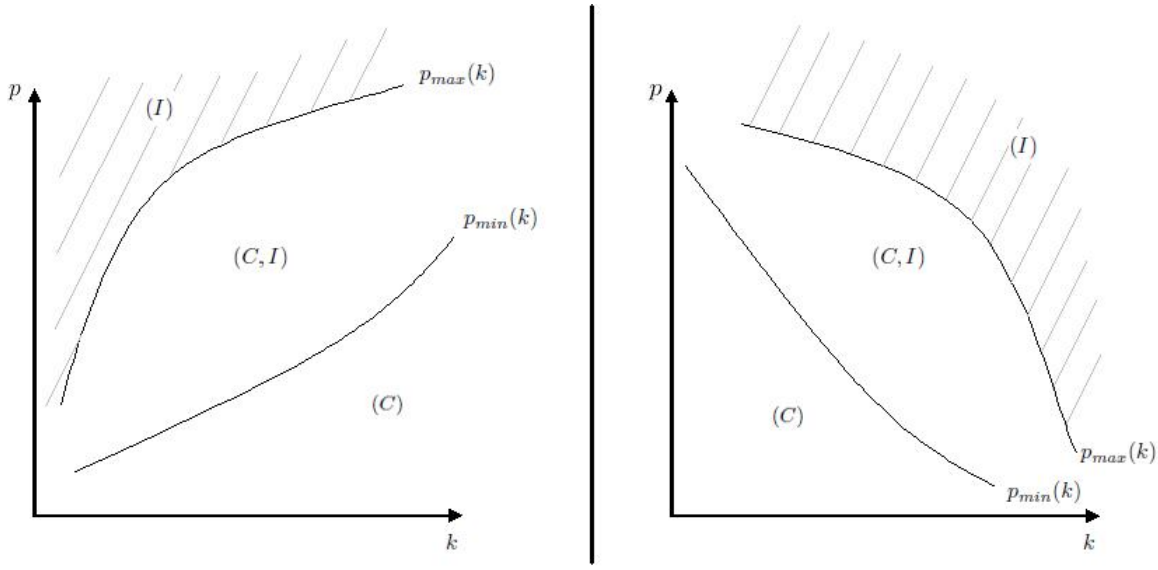

Figure 2: Patterns of specialization in the (k,p) plan when (left) the consumption sector is relatively more capital-intensive and when (right) the investment sector is relatively more capital-intensive.

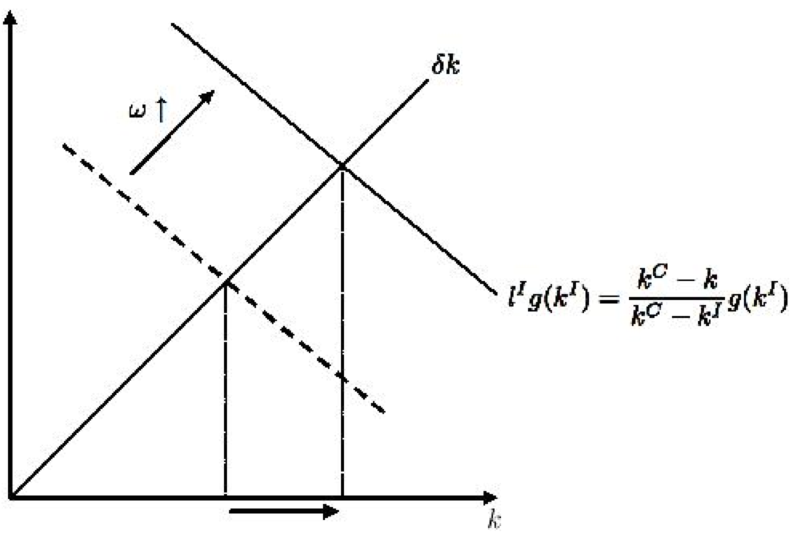

Figure 3: The determination for each $\omega \in (0,\widetilde{\omega})$ of the capital-labor ratio k inducing zero net investment for the case the consumption sector is relatively more capital-intensive.

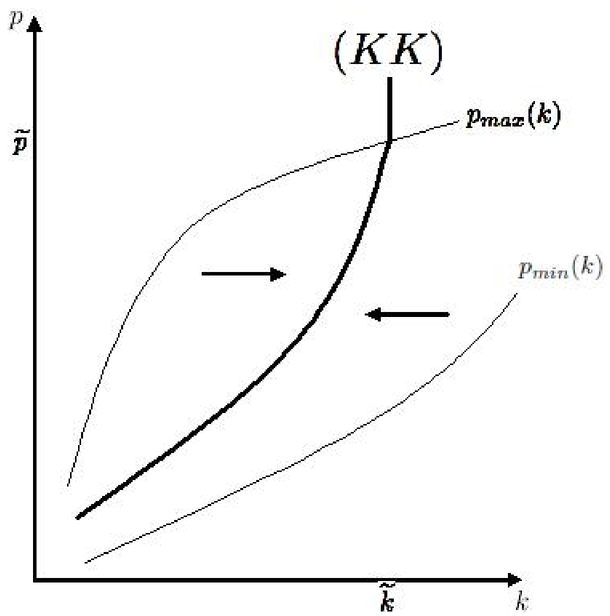

Figure 4: The $\{k_{t+1} = k_t\}$ locus when the consumption sector is relatively more capital-intensive.

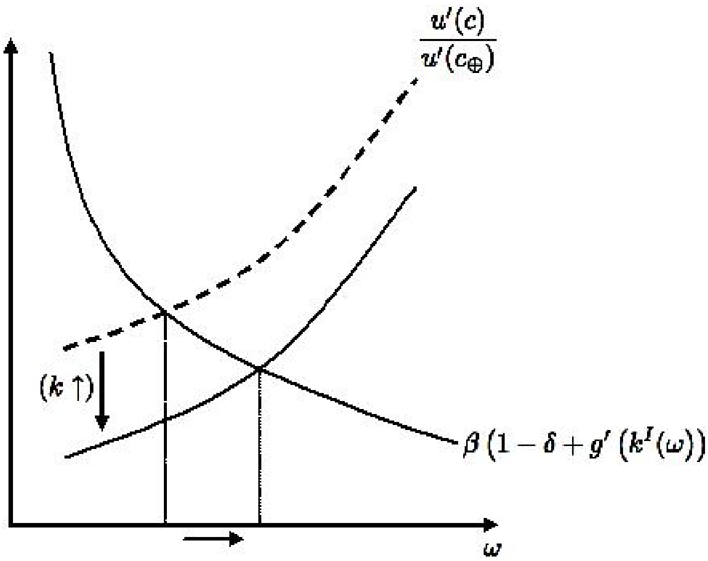

Figure 5: Determination of the $\omega^{PP}(k)$ wage-rental ratio.

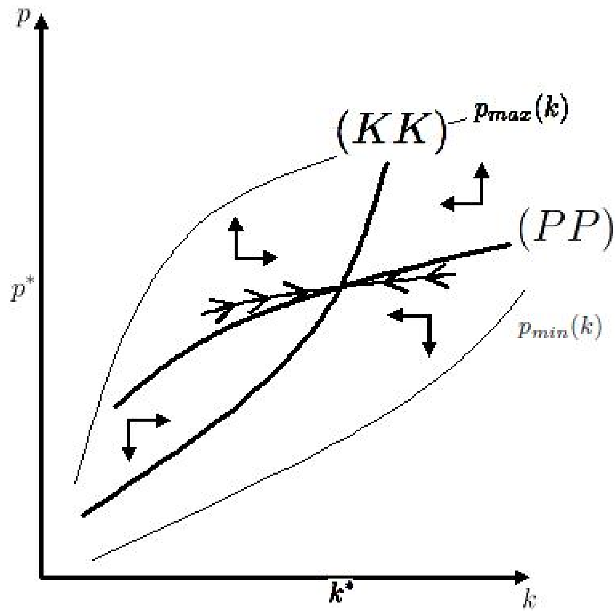

Figure 6: The local phase diagram around quasi-golden rule steady-state when the consumption sector is relatively more capital- intensive.

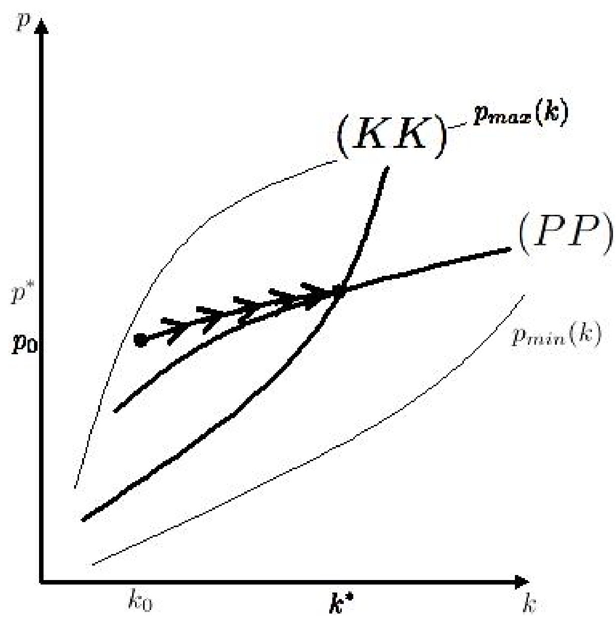

Figure 7: The optimal path for an initially underdeveloped economy ($k_0 < k^*$) when the consumption sector is relatively more capital-intensive.

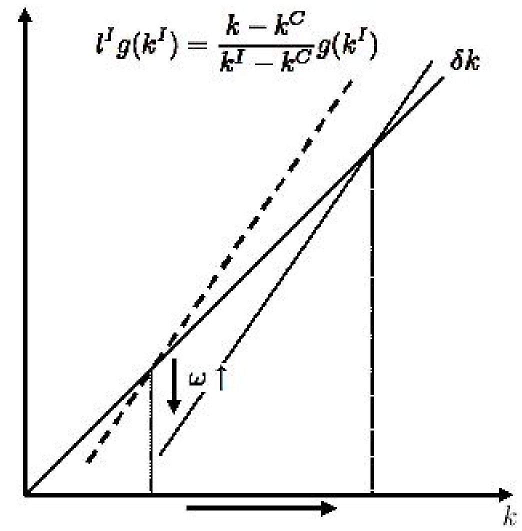

Figure 8: The determination for each $\omega \in (0, \widetilde{\omega})$ of the capital-labor ratio k inducing zero net investment for the case the investment sector is relatively more capital-intensive.

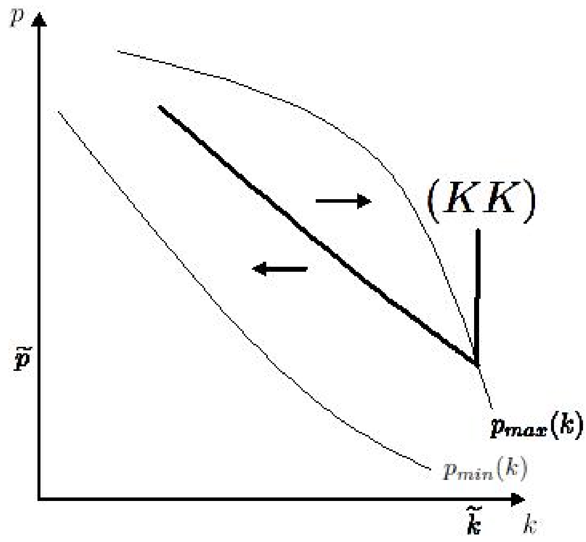

Figure 9: The $\{k_{t+1} = k_t\}$ locus when the investment sector is relatively more capital-intensive.

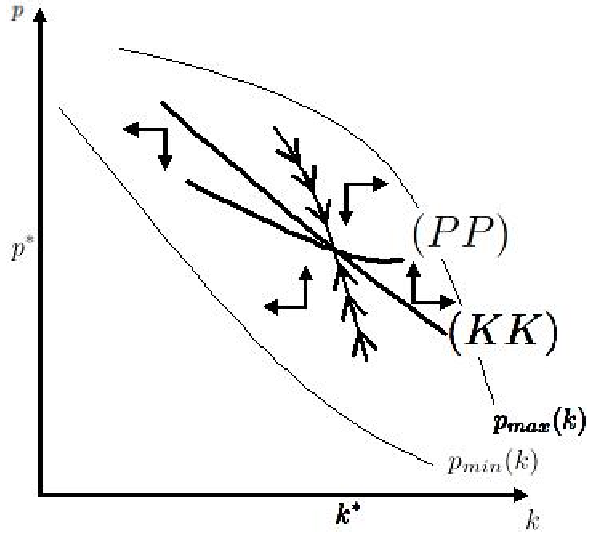

Figure 10: The local phase diagram around quasi-golden rule steady-state when the investment sector is relatively more capital-intensive.

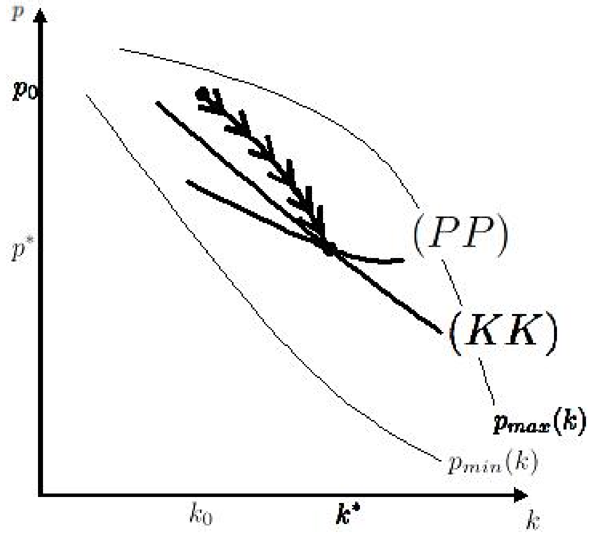

Figure 11: The optimal path for an initially underdeveloped economy ($k_0 < k^*$) when the investment sector is relatively more capital-intensive.

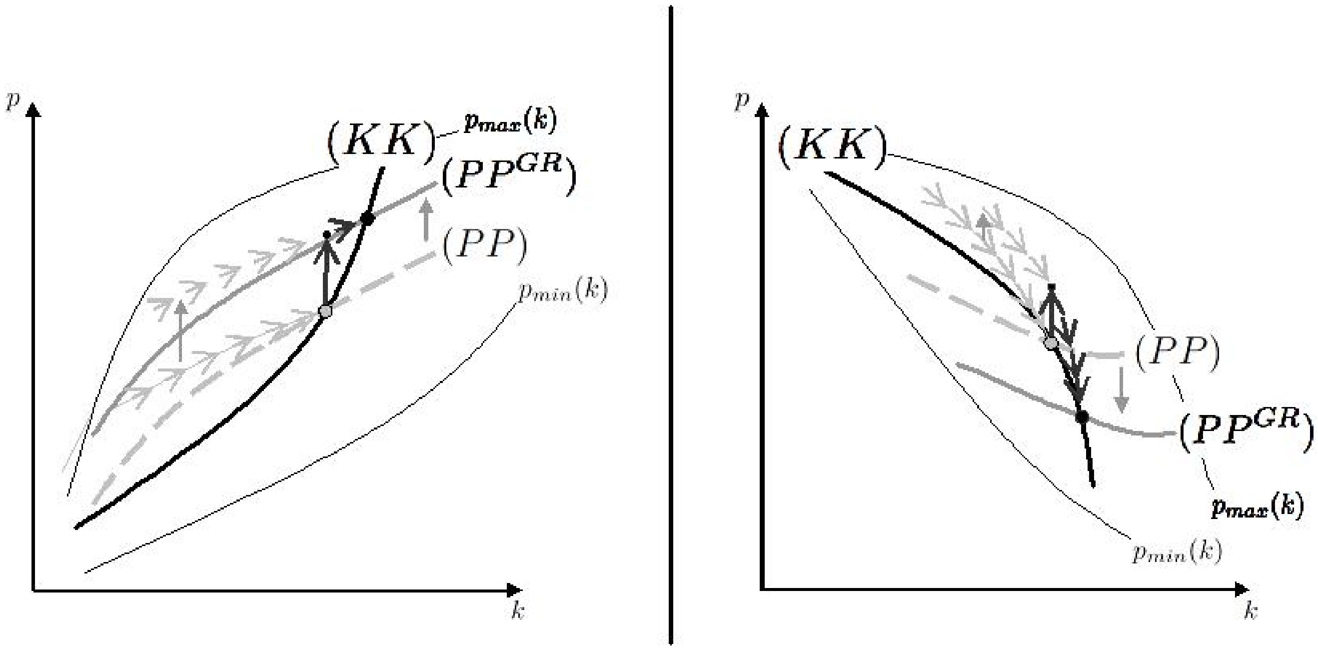

Figure 12: The dynamic effect of a '$\theta \to 0$' shock when (left) the consumption sector is relatively more capital-intensive and when (right) the investment sector is relatively more capital-intensive.

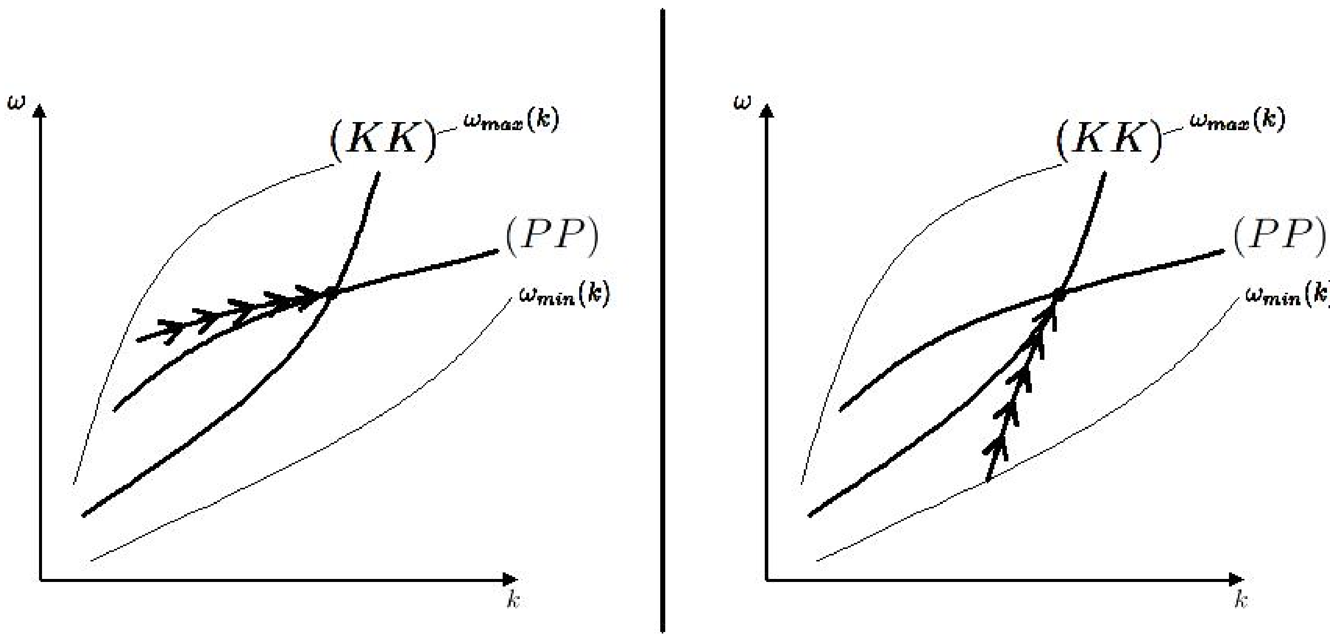

Figure 13: The optimal paths in the (k, ω) plan when (left) the consumption sector is relatively more capital-intensive and when (right) the investment sector is relatively more capital-intensive.

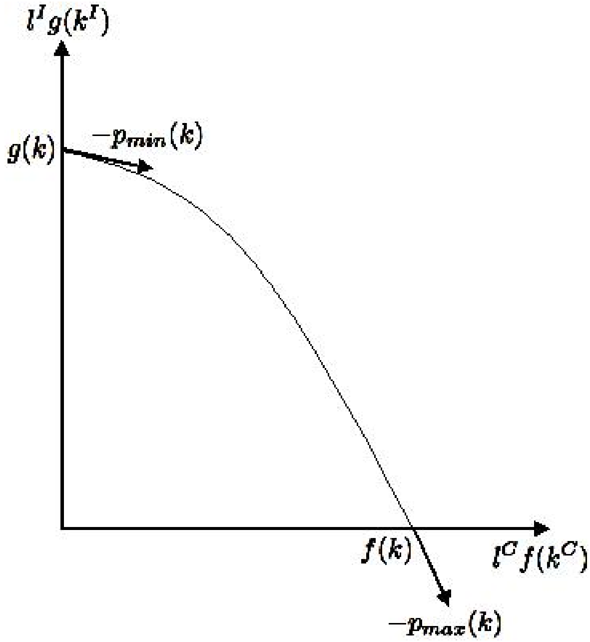

Figure 14: The production possibility frontier for some $k > 0$.

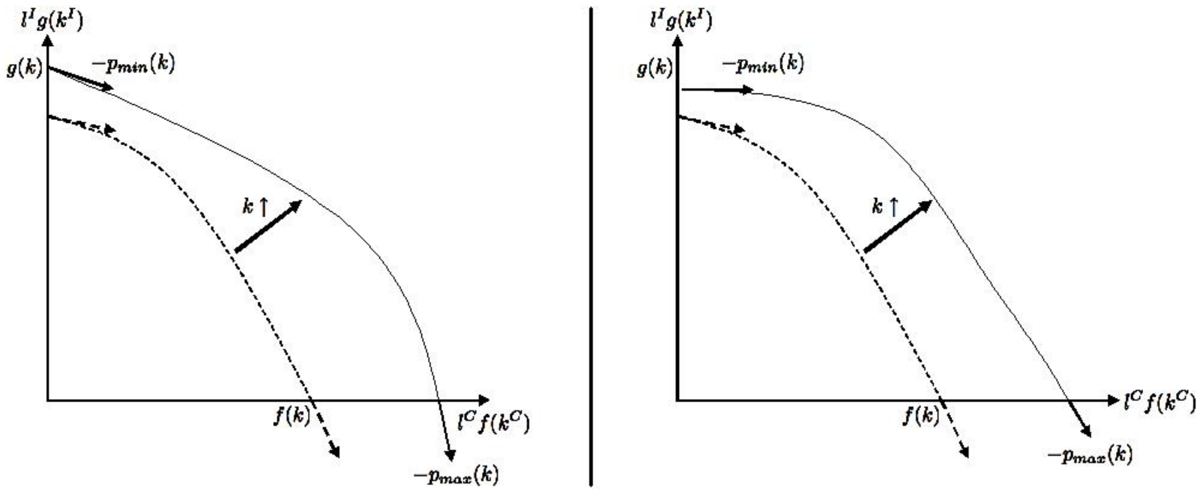

Figure 15: Shifts of the production possibility frontier when k increases. At the left for the case where the consumption sector is more capital-intensive, at the right for the case where the investment sector is more capital-intensive. In the former case, limit prices tend to increase while in the latter case, they tend to decrease.

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